

D'Alembert's Ratio Test of Convergence of Series

In this article we will formulate the D' Alembert's Ratio Test on convergence of a series.

Let's start.

Statement of D'Alembert Ratio Test

A series $\sum u_n$ of positive terms is convergent if from and after some fixed term $\frac{u_{n+1}}{u_n} < r < 1$, where r is a fixed number. The series is divergent if $\frac{u_{n+1}}{u_n} > 1$ from and after some fixed term.

D'Alembert's Test is also known as the **ratio test of convergence of a series**.

Theorem

Let $\sum_{n=1}^{\infty} a_n$ be a series of real numbers in R , or a series of complex numbers in C .

Let the sequence a_n satisfy:

$$\lim_{n \rightarrow \infty} \frac{a_{n+1}}{a_n} = l$$

- If $l > 1$, then $\sum_{n=1}^{\infty} a_n$ diverges.
- If $l < 1$, then $\sum_{n=1}^{\infty} a_n$ converges absolutely.

Definitions for Generally Interested Readers

(Definition 1) An infinite series $\sum u_n$ i.e. $u_1 + u_2 + u_3 + \dots + u_n$ is said to be **convergent** if S_n , the sum of its first n terms, tends to a finite limit S as n tends to infinity.

We call S the sum of the series, and write $S = \lim_{n \rightarrow \infty} S_n$.

Thus an infinite series $\sum u_n$ converges to a sum S , if for any given positive number ϵ , however small, there exists a positive integer n_0 such that $|S_n - S| < \epsilon$ for all $n \geq n_0$.

(Definition 2)

If $S_n \rightarrow \pm\infty$ as $n \rightarrow \infty$, the series is said to be **divergent**.

Thus, $\sum u_n$ is said to be divergent if for every given positive number λ , however large, there exists a positive integer n_0 such that $|S_n| > \lambda$ for all $n \geq n_0$.

(Definition 3)

If S_n does not tend to a finite limit, or to plus or minus infinity, the series is called **oscillatory**.

Proof & Discussions on Ratio Test

Let a series be $u_1 + u_2 + u_3 + \dots$. We assume that the above inequalities are true.

- From the first part of the statement:

$$\frac{u_2}{u_1} < r, \frac{u_3}{u_2} < r \dots \text{where } r < 1.$$

$$\begin{aligned} \text{Therefore } u_1 + u_2 + u_3 + \dots &= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_1} + \dots \right) \\ &= u_1 \left(1 + \frac{u_2}{u_1} + \frac{u_3}{u_2} \times \frac{u_2}{u_1} + \dots \right) \\ &< u_1 (1 + r + r^2 + \dots) \end{aligned}$$

$$\text{Therefore, } \sum u_n < u_1 (1 + r + r^2 + \dots)$$

$$\text{or, } \sum u_n < \lim_{n \rightarrow \infty} \frac{u_1(1 - r^n)}{1 - r}$$

Since $r < 1$, therefore as $n \rightarrow \infty$, $r^n \rightarrow 0$

$$\text{therefore } \sum u_n < \frac{u_1}{1 - r} = k \text{ say, where } k \text{ is a fixed number.}$$

Therefore $\sum u_n$ is convergent.

- Since, $\frac{u_{n+1}}{u_n} > 1$ then, $\frac{u_2}{u_1} > 1, \frac{u_3}{u_2} > 1 \dots$

Therefore $u_2 > u_1, u_3 > u_2 > u_1, u_4 > u_3 > u_2 > u_1$ and so on.

Therefore $\sum u_n = u_1 + u_2 + u_3 + \dots + u_n > nu_1$. By taking n sufficiently large, we see that nu_1 can be made greater than any fixed quantity.

Hence the series is divergent.

Academic Proof

From the statement of the theorem, it is necessary that $\forall n : a_n \neq 0$; otherwise $\frac{a_{n+1}}{a_n}$

is not defined.

Here, $\frac{a_{n+1}}{a_n}$ denotes either the absolute value of $\frac{a_{n+1}}{a_n}$, or the complex modulus of $\frac{a_{n+1}}{a_n}$.

Absolute Convergence

Suppose $l < 1$.

Let us take $\epsilon > 0$ such that $l + \epsilon < 1$.

Then:

$$\exists N : \forall n > N : \frac{a_n}{a_{n-1}} < l + \epsilon$$

Thus:

$$(a_n)(=) \left(\frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \dots \frac{a_{N+2}}{a_{N+1}} a_{N+1} \right)$$

$$(<)(l + \epsilon^{n-N+1} a_{N+1})$$

By Sum of Infinite Geometric Progression, $\sum_{n=1}^{\infty} l + \epsilon^n$ converges.

So by the the corollary to the comparison test, it follows that $\sum_{n=1}^{\infty} a_n$ converges

absolutely too.



Divergence

Suppose $l > 1$.

Let us take $\epsilon > 0$ small enough that $l - \epsilon > 1$.

Then, for a sufficiently large N , we have:

$$(a_n)(=) \left(\frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} \dots \frac{a_{N+2}}{a_{N+1}} a_{N+1} \right)$$

$$(>)(l - \epsilon^{n-N+1} a_{N+1})$$

But $l - \epsilon^{n-N+1} a_{N+1} \rightarrow \infty$ as $n \rightarrow \infty$.

So $\sum_{n=1}^{\infty} a_n$ diverges.



Comments

- When $\frac{u_{n+1}}{u_n} = 1$, the test fails.
- Another form of the test– The series $\sum u_n$ of positive terms is convergent if $\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} > 1$ and divergent if $\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} < 1$.

One should use this form of the test in the practical applications.

An Example

Verify whether the infinite series $\frac{x}{1.2} + \frac{x^2}{2.3} + \frac{x^3}{3.4} + \dots$ is convergent or divergent.

Solution

We have $u_{n+1} = \frac{x^{n+1}}{(n+1)(n+2)}$ and $u_n = \frac{x^n}{n(n+1)}$

Therefore $\lim_{n \rightarrow \infty} \frac{u_n}{u_{n+1}} = \lim_{n \rightarrow \infty} \left(1 + \frac{2}{n}\right) \frac{1}{x} = \frac{1}{x}$

Hence, when $1/x > 1$, i.e., $x < 1$, the series is convergent and when $x > 1$ the series is divergent.

When $x=1$, $u_n = \frac{1}{n(n+1)} = \frac{1}{n^2} (1 + 1/n)^{-1}$

or, $u_n = \frac{1}{n^2} (1 - \frac{1}{n} + \frac{1}{n^2} - \dots)$

Take $\frac{1}{n^2} = v_n$ Now $\lim_{n \rightarrow \infty} \frac{u_n}{v_n} = 1$, a non-zero finite quantity.

But $\sum v_n = \sum \frac{1}{n^2}$ is convergent.

Hence, $\sum u_n$ is also convergent.